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(50.) Equal ellipses are drawn within a polygon as in problem 49, and equal circles are drawn in the area common to two consecutive ellipses. Given the major and minor axes, find the diameter of the circles.*

As an appendix of this book, the author Murata described the minute demonstrations of the solutions of the sixth and eleventh problems of Ajima's Fukyū-Sampō in which the solutions of all problems were not demonstrated.

I will conclude this paper by adding that ellipsoids which were treated by Japanese mathematicians of the old school were spheroids only, called Chōryūen or elongated solid circle.

Tokyo, *December 1, 1905.*

ON THE FUNCTIONS WHICH HAVE A GIVEN ALGEBRAICAL ADDITION THEOREM.

By M. KABA, Tokyo, Japan.

Let $f(x)$ denote the uniform analytical function possessing a given algebraical addition theorem, that is,

$$f(x+y)=F[f(x),f(y)]\ldots(1).$$

It is evident that the function satisfies the equation

$$f(2x)=F[f(x),f(x)]\ldots(2).$$

In the first place we prove conversely that the uniform analytical function $\varphi(x)$ which satisfies equation (2) will satisfy equation (1). Set

$$\varphi(x+y)-F[\varphi(x),\ \varphi(y)]=F_1(x,\ y).$$

If we use $x+h$ instead of y and develop the function F_1 in a power series of h by Maclaurin's theorem, all the coefficients will be zero. In fact, as the function F is symmetrical in $\varphi(x)$ and $\varphi(y)$, we have the following:

$$\left[\frac{dF}{dh}\right]_{h=0}=\frac{1}{2}\frac{d}{dx}F[\varphi(x),\ \varphi(x)],\qquad \left[\frac{d^2F}{dh^2}\right]_{h=0}=\frac{1}{2^2}\frac{d^2}{dx^2}F[\varphi(x),\ \varphi(x)],\ \ldots$$

And also we have the following:

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*In justice to the author and to Editor Dickson, I wish to state that I changed the enunciation of all the problems unaccompanied by diagrams, to avoid making so many wood-cuts. Ed. F.

$$\left[\frac{d\,\varphi(2x+h)}{dh}\right]_{h=0}=\frac{1}{2}\frac{d\varphi(2x)}{dx},\quad \left[\frac{d^2\varphi(2x+h)}{dh^2}\right]_{h=0}=\frac{1}{2^2}\frac{d^2\varphi(2x)}{dx},\ ..$$

Therefore we have:

$$(F_1)_{h=0}=\varphi(2x)-F[\varphi(x),\ \varphi(x)],$$

$$\left[\frac{dF_1}{dh}\right]_{h=0}=\frac{1}{2}\frac{d}{dx}\Big[\varphi(2x)-F[\varphi(x),\ \varphi(x)]\Big],$$

$$\left[\frac{d^2\,F_1}{dh^2}\right]_{h=0}=\frac{1}{2^2}\frac{d^2}{dx^2}\Big[\varphi(2x)-F[\varphi(x),\ \varphi(x)]\Big],\ \dots$$

But as $\varphi(x)$ satisfies the equation (2), we have:

$$(F_1)_{h=0}=0,\quad \left[\frac{dF_1}{dh}\right]_{h=0}=0,\quad \left[\frac{d^2\,F_1}{dh^2}\right]_{h=0}=0,\ \dots$$

In the second place, we have to prove that there is a single uniform analytical function which satisfies the equation (2).

Let us suppose that the function $f(x)+\theta(x)$ satisfies the equation (2), where $f(x)$ satisfies (1) and therefore satisfies (2). From what we have said above, we know that $f(x)+\theta(x)$ possesses an addition theorem. But $f(x)$ possesses the given addition theorem (1). Therefore $f(x)+\theta(x)$ must be one of the three functions: Algebraical functions, simply periodical functions, doubly periodical functions. Hence $\theta(x)$ also must be one of these three functions, and therefore it possesses an addition theorem; that is

$$\theta(x+y)=F_1[\theta(x),\ \theta(y)],$$

and $\theta(x)$ has satisfied the eqnation

$$\theta(2x)=F_1[\theta(x),\ \theta(x)].$$

By the foregoing hypothesis, we have

$$f(2x)+\theta(2x)=F[f(x)+\theta(x),\ f(x)+\theta(x)],$$

that is,

$$F[f(x),f(x)]+F_1[\theta(x),\ \theta(x)]=F[f(x)+\theta(x),f(x)+\theta(x)].$$

Let us use y instead of $f(x)$ and z instead of $\theta(x)$, then we shall get the following identity:

$$F_1(z)=F(y+z)-F(y)$$

where F and F_1 are the algebraical functions of y and z. But as the first member of the identity is independent of y,

$$F'(y+z)-F'(y)=0.$$

Then $F'(y)$ must be a constant and $F(y)$ will have the form

$$F(y)=ay+b.$$

Also $f(x)$ and $\theta(x)$ will have the relations, respectively:

$$f(2x)=af(x)+b, \quad \theta(2x)=a\theta(x).$$

Therefore both $f(x)$ and $\theta(x)$ must be constants. Hence we have the following theorem:

The uniform analytical function satisfying a given addition theorem is the only one which satisfies the equation acquired by making $x=y$ on the addition theorem.

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DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

The following note will be of value, since it calls attention to the fact that unless the law of a series is given, the *definite* determination of the series is impossible. If only a certain number of terms of a series is given and not the law of the series, the actual law of the series can in no case be really determined. All that can be done is to find the *simplest law* which the few given terms will obey. The solutions referred to may be justified on the ground that they were attempts to find this *simplest law*. ED. F.

NOTE ON THE SOLUTION OF PROBLEM 266, IN AUGUST-SEPTEMBER MONTHLY.

By **CLARENCE E. COMSTOCK**, Bradley Polytechnic Institute.

Find the nth term and the sum of n terms of the series $1+3+7+17+\dots$

One writer assumes without the least excuse for such an assumption that it is an arithmetical series of the third order and proceeds by the method of finite differences, getting a result in accord with his assumption. Five terms of his series are $1+3+7+17+37+\dots$

Another writer assumes with a little more plausible excuse that it is a recurring series of the second order and, of course, gets a result in accord with his assumption. Five terms of his series are $1+3+7+17+41+\dots$

Suppose I make the assumption that it is an arithmetical series of the fourth order. I can then build up